Illustrating Kinematics With Motion Graphs

CoA Physics

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Abstract

In this note, we will collect the examples of motion graphs—plots of $x(t)$, $v(t)$, and/or $a(t)$—given in the textbook, OpenStax *University Physics Volume 1*, and we will add motion graphs on additional kinematics examples from the textbook. We will also summarize additional examples and drawing of motion graphs from the lecture.

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I. INTRODUCTION

Core ideas in kinematics can be stated in these two differential equations,

$$v = dx/dt \qquad \text{and} \qquad a = dv/dt,$$

along with the idea that $x(t)$ represents position (i.e. location) of an object as a function of time.

Many examples are needed, however, in learning to apply these core ideas, both in artificial classroom situations and in more realistic physical setups. Visual problem-solving tools are useful in working through these examples. They allow specification of the example with more detail than we can with the English language, and they help avoid overreliance on memorized formulas, which might work for a few memorized problems but are too inflexible to be useful generally.

Motion graphs—2D plots of $x(t)$, $v(t)$, and $a(t)$, with time t on horizontal axis and one of the kinematic quantities on vertical axis—are the visual problem-solving tools for kinematics problems, and we will go through some examples of their usage.

II. OPENSTAX EXAMPLES

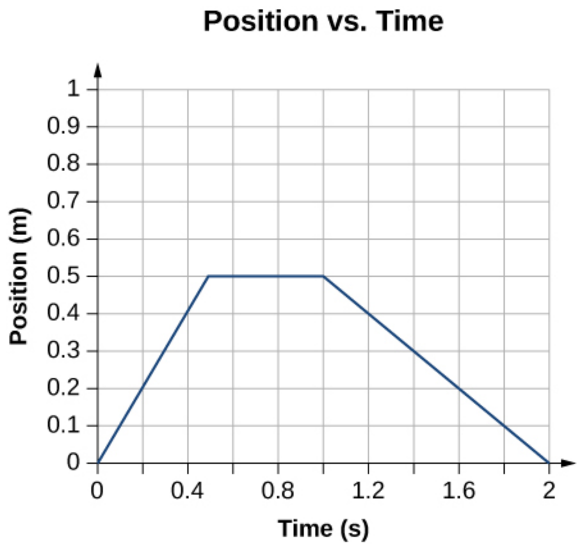
Motion graphs are covered in OpenStax *University Physics Volume 1* with some examples. We will include those examples in this section, and supplement additional textbook examples with our motion graphs. OpenStax *University Physics Volume 1* is licensed under CC-BY 4.0 and is available free of charge at https://openstax.org/details/books/university-physics-volume-1.

A. OpenStax Example 3.2

The textbook example is reproduced verbatim below.

Finding Velocity from a Positon-Versus-Time Graph

Given the position-versus-time graph of Figure 3.7, find the velocity-versus-time graph.



[**Figure 3.7** The object starts out in the positive direction, stops for a short time, and then reverses direction, heading back toward the origin. Notice that the object comes to rest instantaneously, which would require an infinite force. Thus, the graph is an approximation of motion in the real world. (The concept of force is discussed in Newton's Laws of Motion.)]

Strategy — The graph contains three straight lines during three time intervals. We find the velocity during each time interval by taking the slope of the line using the grid.

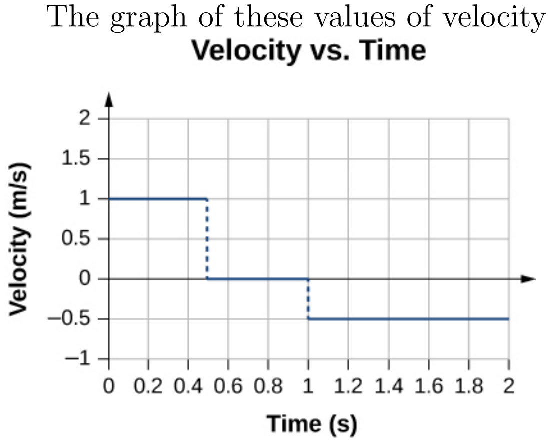
Solution

Time interval 0 s to 0.5 s: $\overline{v} = \frac{\Delta x}{\Delta t} = \frac{0.5\ \text{m} - 0.0\ \text{m}}{0.5\ \text{s} - 0.0\ \text{s}} = 1.0\ \text{m/s}$

Time interval 0.5 s to 1.0 s: $\overline{v} = \frac{\Delta x}{\Delta t} = \frac{0.5\ \text{m} - 0.5\ \text{m}}{1.0\ \text{s} - 0.5\ \text{s}} = 0.0\ \text{m/s}$

Time interval 1.0 s to 2.0 s: $\overline{v} = \frac{\Delta x}{\Delta t} = \frac{0.0\ \text{m} - 0.5\ \text{m}}{2.0\ \text{s} - 1.0\ \text{s}} = -0.5\ \text{m/s}$

The graph of these values of velocity versus time is shown in Figure 3.8.



[**Figure 3.8** The velocity is positive for the first part of the trip, zero when the object is stopped, and negative when the object reverses direction.]

Significance — During the time interval between 0 s and 0.5 s, the object's position is moving away from the origin and the position-versus-time curve has a positive slope. At any point along the curve during this time interval, we can find the instantaneous velocity by taking its slope, which is +1 m/s, as shown in Figure 3.8. In the subsequent time interval, between 0.5 s and 1.0 s, the position doesn't change and we see the slope is zero. From 1.0 s to 2.0 s, the object is moving back toward the origin and the slope is −0.5 m/s. The object has reversed direction and has a negative velocity.

B. OpenStax Example 3.6

The textbook example is reproduced verbatim below. Note that this example illustrates a non-constant acceleration motion, as acceleration $a(t)$ is a function of time.

Calculating Instantaneous Acceleration

A particle is in motion and is accelerating. The functional form of the velocity is $v(t) = (20 \text{ m/s})t - (5 \text{ m/s}^2)t^2$.

a. Find the functional form of the acceleration.

b. Find the instantaneous velocity at $t = 1,\ 2,\ 3,$ and 5 s.

c. Find the instantaneous acceleration at $t = 1,\ 2,\ 3,$ and 5 s.

d. Interpret the results of (c) in terms of the directions of the acceleration and velocity vectors.

Strategy — We find the functional form of acceleration by taking the derivative of the velocity function. Then, we calculate the values of instantaneous velocity and acceleration from the given functions for each. For part (d), we need to compare the directions of velocity and acceleration at each time.

Solution

a. $a(t) = \frac{dv(t)}{dt} = (20 \text{ m/s}^2) - (10 \text{ m/s}^3)t$

b. $v(1 \text{ s}) = 15$ m/s, $v(2 \text{ s}) = 20$ m/s, $v(3 \text{ s}) = 15$ m/s, $v(5 \text{ s}) = -25$ m/s

c. $a(1 \text{ s}) = 10 \text{ m/s}^2$, $a(2 \text{ s}) = 0 \text{ m/s}^2$, $a(3 \text{ s}) = -10 \text{ m/s}^2$, $a(5 \text{ s}) = -30 \text{ m/s}^2$

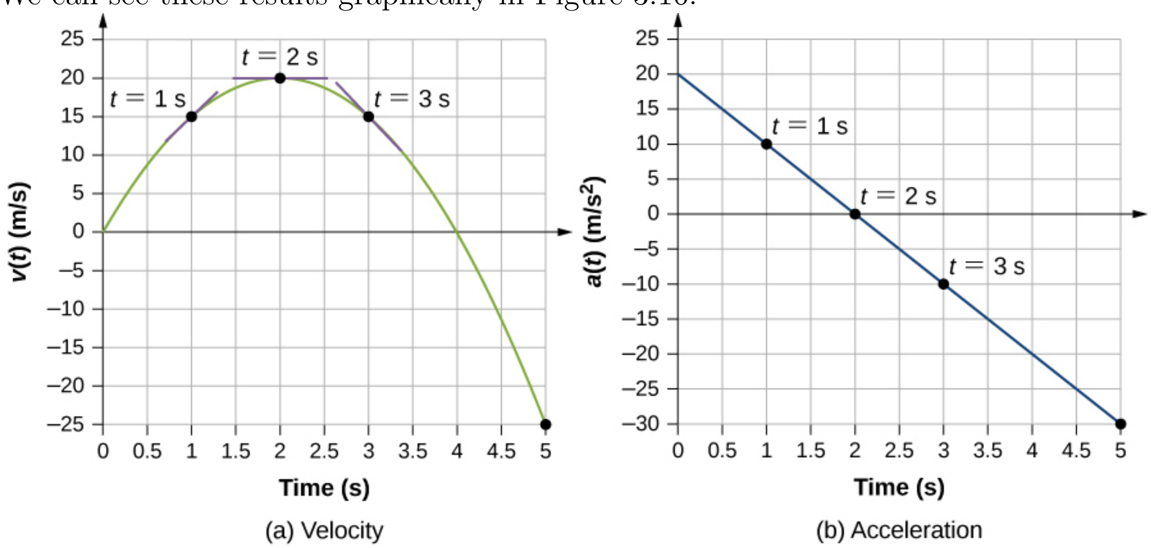
d. At $t = 1$ s, velocity $v(1 \text{ s}) = 15$ m/s is positive and acceleration is positive, so both velocity and acceleration are in the same direction. The particle is moving faster.

At $t = 2$ s, velocity has increased to $v(2\text{ s}) = 20$ m/s, where it is maximum, which corresponds to the time when the acceleration is zero. We see that the maximum velocity occurs when the slope of the velocity function is zero, which is just the zero of the acceleration function.

At $t = 3$ s, velocity is $v(3\text{ s}) = 15$ m/s and acceleration is negative. The particle has reduced its velocity and the acceleration vector is negative. The particle is slowing down.

At $t = 5$ s, velocity is $v(5\text{ s}) = -25$ m/s and acceleration is increasingly negative. Between the times $t = 3$ s and $t = 5$ s the particle has decreased its velocity to zero and then become negative, thus reversing its direction. The particle is now speeding up again, but in the opposite direction.

We can see these results graphically in Figure 3.16.



[**Figure 3.16** (a) Velocity versus time. Tangent lines are indicated at times 1, 2, and 3 s. The slopes of the tangent lines are the accelerations. At $t = 3$ s, velocity is positive. At $t = 5$ s, velocity is negative, indicating the particle has reversed direction. (b) Acceleration versus time. Comparing the values of accelerations given by the black dots with the corresponding slopes of the tangent lines (slopes of lines through black dots) in (a), we see they are identical.]

Significance — By doing both a numerical and graphical analysis of velocity and acceleration of the particle, we can learn much about its motion. The numerical analysis complements the graphical analysis in giving a total view of the motion. The zero of the acceleration function corresponds to the maximum of the velocity in this example. Also in this example, when acceleration is positive and in the same direction as velocity, velocity increases. As acceleration tends toward zero, eventually becoming negative, the velocity reaches a maximum, after which it starts decreasing. If we wait long enough, velocity also becomes negative, indicating a reversal of direction. A real-world example of this type of motion is a car with a velocity that is increasing to a maximum, after which it starts slowing down, comes to a stop, then reverses direction.

C. OpenStax Example 3.3, with added motion graphs

Example 3.3, **Instantaneous Velocity Versus Average Velocity** illustrates and compares instantaneous velocity and average velocity. While the textbook example stops with the numerical comparison, we will add the motion graphs and see how this visual representation matches with the numerical results in the textbook example.

From the example: The position of a particle is given by

$$x(t) = (3.0\ \mathrm{m/s})t + (0.5\ \mathrm{m/s^3})t^3,$$

and the instantaneous velocity at $t = 2.0$ s and the average velocity between 1.0 s and 3.0 s are asked for. (Answers in the textbook: $v(2.0\ \mathrm{s}) = 3.0\ \mathrm{m/s} + (1.5\ \mathrm{m/s^3})\,(2.0\ \mathrm{s})^2 = 9\ \mathrm{m/s}$, and $\overline{v} = 9.5$ m/s, between 1.0 s and 3.0 s.)

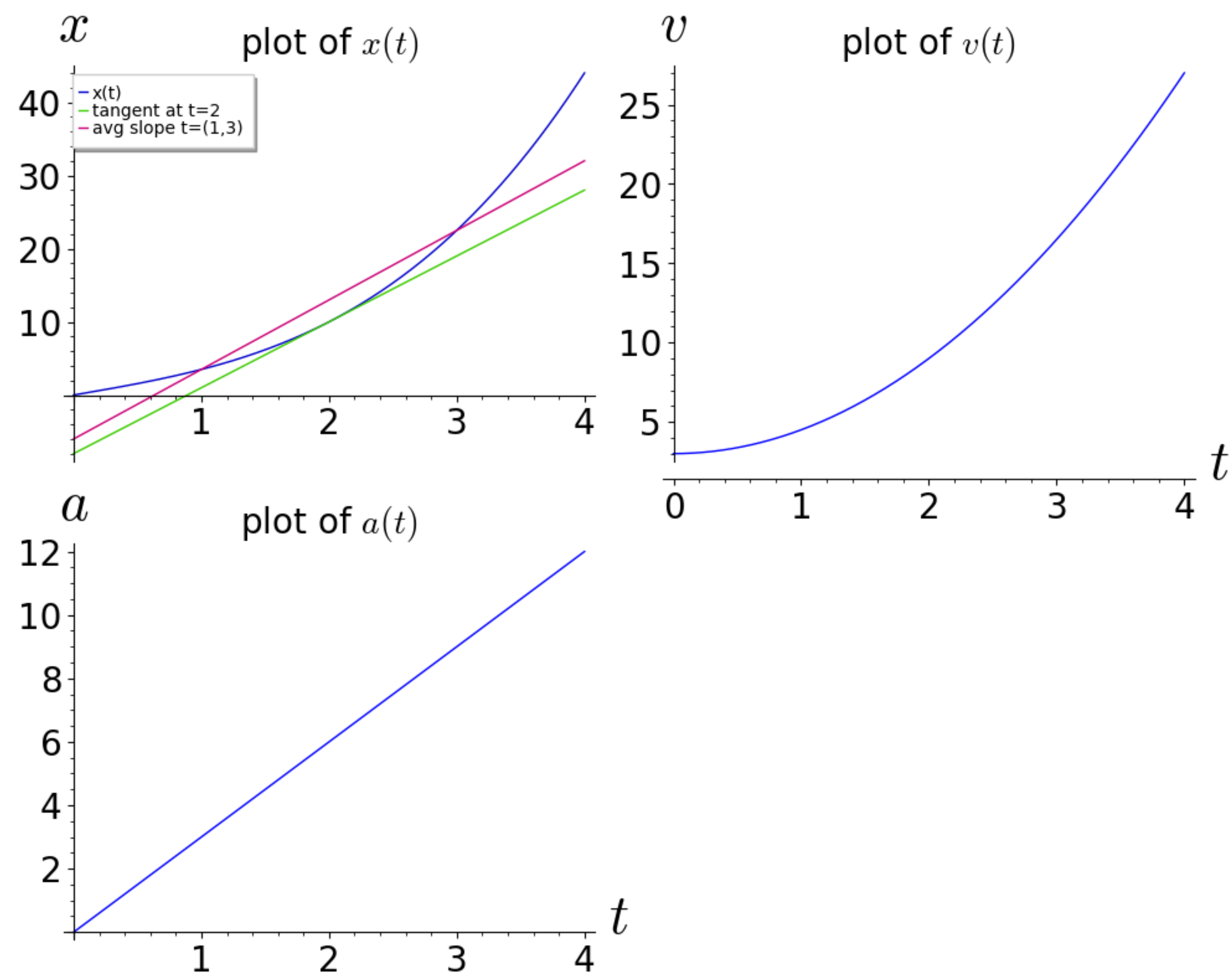
The example also gives the instantaneous velocity as a function of time, from $v(t) = dx/dt$,

$$v(t) = \frac{dx(t)}{dt} = (3.0\ \mathrm{m/s}) + \left(1.5\ \mathrm{m/s^3}\right) t^2.$$

We can go one step further and also get an expression for instantaneous acceleration as a function of time, from $a(t) = dv/dt$,

$$a(t) = \frac{dv(t)}{dt} = (0\ \mathrm{m/s}) + \left(3.0\ \mathrm{m/s^3}\right) t.$$

Let's graph these functions $x(t)$, $v(t)$, and $a(t)$ over some time interval and see how they compare. On the plot of $x(t)$ below, two additional lines are shown to illustrate the two values of velocity calculated in the example, the tangent line for the instantaneous velocity at $t = 2$ s, and the line between the two points $(1\ \mathrm{s},\ 3.5\ \mathrm{m})$ and $(3\ \mathrm{s},\ 22.5\ \mathrm{m})$ for the average velocity between $t = 1$ s and $t = 3$ s.



Caution is needed as we look at these graphs. Superficially, the plot of $x(t)$ appears similar to the plot of $v(t)$ in this example. In most realistic examples we will look at, we won't see this kind of similarity. To see why in the most general cases $v(t)$ does not look similar to $x(t)$, remember that they are related through a differentiation relationship; $v(t)$ is obtained by taking a time derivative of $x(t)$. In calculus, few functions retain the same form when you take its derivative, the one exception being the exponential function. (And unfortunately for this example, high-order polynomial can visually appear similar to an exponential.)

D. OpenStax Example 3.17, with added $a(t)$ graph

The textbook example is first reproduced verbatim below. After the "Significance" section of the textbook example, we will add the $a(t)$ graph and point out salient features of motion graphs.

Motion of a Motorboat

A motorboat is traveling at a constant velocity of 5.0 m/s when it starts to accelerate opposite to the motion to arrive at the dock. Its acceleration is $a(t) = -\left(\frac{1}{4}\ \text{m/s}^3\right) t$. (a) What is the velocity function of the motorboat? (b) At what time does the velocity reach zero? (c) What is the position function of the motorboat? (d) What is the displacement of the motorboat from the time it begins to accelerate opposite to the motion to when the velocity is zero? (e) Graph the velocity and position functions.

Strategy — (a) To get the velocity function we must integrate and use initial conditions to find the constant of integration. (b) We set the velocity function equal to zero and solve for t. (c) Similarly, we must integrate to find the position function and use initial conditions to find the constant of integration. (d) Since the initial position is taken to be zero, we only have to evaluate the position function at the time when the velocity is zero.

Solution

We take $t = 0$ to be the time when the boat starts to accelerate opposite to the motion.

a. From the functional form of the acceleration we can solve Equation 3.18 to get $v(t)$:

$$v(t) = \int a(t)\ dt + C_1 = \int -\left(\frac{1}{4}\ \text{m/s}^3\right) t\ dt + C_1 = -\left(\frac{1}{8}\ \text{m/s}^3\right) t^2 + C_1.$$

At $t = 0$ we have $v(0) = 5.0\ \text{m/s} = 0 + C_1$, so $C_1 = 5.0\ \text{m/s}$ or $v(t) = 5.0\ \text{m/s} - \left(\frac{1}{8}\ \text{m/s}^3\right) t^2$.

b. $v(t) = 0 = 5.0\ \text{m/s} - \left(\frac{1}{8}\ \text{m/s}^3\right) t^2 \Rightarrow t = 6.3\ \text{s}$

c. Solve Equation 3.19:

$$x(t) = \int v(t)\ dt + C_2 = \int \left(5.0\ \text{m/s} - \left(\frac{1}{8}\ \text{m/s}^3\right) t^2\right)\ dt + C_2 = (5.0\ \text{m/s})t - \left(\frac{1}{24}\ \text{m/s}^3\right) t^3 + C_2.$$

At $t = 0$, we set $x(0) = 0 = x_0$, since we are only interested in the displacement from when the boat starts to accelerate opposite to the motion. We have

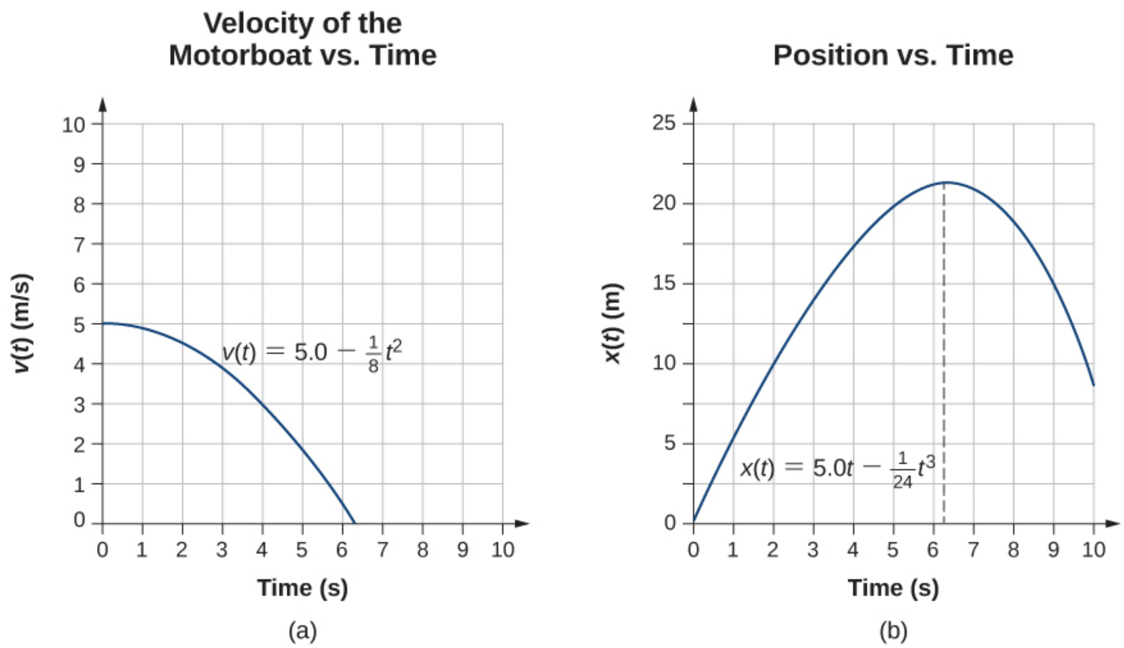
$$x(0) = 0 = C_2.$$

Therefore, the equation for the position is

$$x(t) = (5.0\ \text{m/s})t - \left(\frac{1}{24}\ \text{m/s}^3\right) t^3.$$

d. Since the initial position is taken to be zero, we only have to evaluate the position function at the time when the velocity is zero. This occurs at $t = 6.3$ s. Therefore, the displacement is

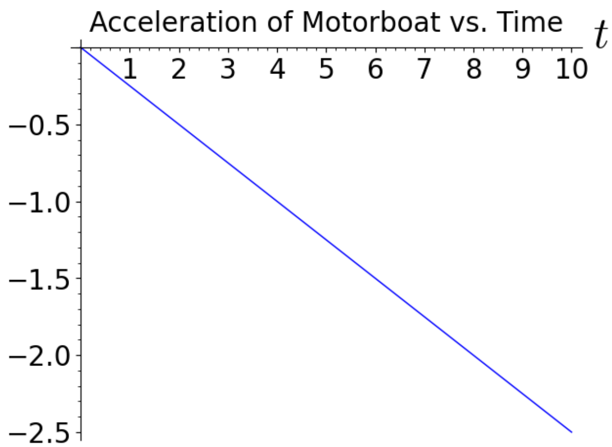
$$x(6.3 \text{ s}) = (5.0 \text{ m/s})(6.3 \text{ s}) - \left(\frac{1}{24} \text{ m/s}^3\right)(6.3 \text{ s})^3 = 21.1 \text{ m}.$$



[**Figure 3.30** (a) Velocity of the motorboat as a function of time. The motorboat decreases its velocity to zero in 6.3 s. At times greater than this, velocity becomes negative—meaning, the boat is reversing direction. (b) Position of the motorboat as a function of time. At $t = 6.3$ s, the velocity is zero and the boat has stopped. At times greater than this, the velocity becomes negative—meaning, if the boat continues to move with the same acceleration, it reverses direction and heads back toward where it originated.]

Significance — The acceleration function is linear in time so the integration involves simple polynomials. In Figure 3.30, we see that if we extend the solution beyond the point when the velocity is zero, the velocity becomes negative and the boat reverses direction. This tells us that solutions can give us information outside our immediate interest and we should be careful when interpreting them.

Additional Motion Graph Notes — Below is the $a(t)$ motion graph from the given function $a(t) = -\left(\frac{1}{4} \text{ m/s}^3\right) t$.



Some features of these motion graphs are worth noting and can be expected in other examples of motion graphs.

- There is little visual similarity between $a(t)$, $v(t)$, and $x(t)$ graphs. With rare exceptions (when acceleration looks like $a(t) = a_0 \exp(\rho t)$, i.e. exponential acceleration), integration or differentiation steps between $a(t)$, $v(t)$, and $x(t)$ won't give you the same function you started with. In comparing motion graphs, instead of visual similarity, look for below points of comparison.

- Slope at different points of $v(t)$ match with values of $a(t)$ at the same time. For example, at $t = 0$, the $v(t)$ curve appears flat, and this matches with the value of $a = 0$ at $t = 0$. At later times, as $a(t)$ gets more negative, $v(t)$ becomes more and more downward-sloping—but not negative, until after about 6 seconds.

- Slope at different points of $x(t)$ match with values of $v(t)$ at the same time. At $t = 0$, $x(t)$ starts with an upward-sloping line, which matches the positive value of $v(t)$ at $t = 0$. This upward slope of $x(t)$ continues until time $t = 6.3$ s, when $v = 0$, and $x(t)$ becomes flat. Note that $x(t)$ will remain positive for quite some time after $v(t)$ turns negative, as $v(t)$ turning negative simply means slope of $x(t)$ will curve downward.

This relationship between the value of $a(t)$ and slope of $v(t)$ (and similar for value of $v(t)$ and slope of $x(t)$) is the derivative relationship, $a(t) = dv/dt$. One can work out the reverse relationship as well, using the relationship between the integral and "area under the curve," but some care is needed to make sure this is done correctly. Specifically, "area under the curve" of $a(t)$ graph will only represent the change of velocity (Δv) over the time interval. This is laid out in some detail in lecture video Derivation of Kinematics Formulas.

III. LECTURE SUMMARY

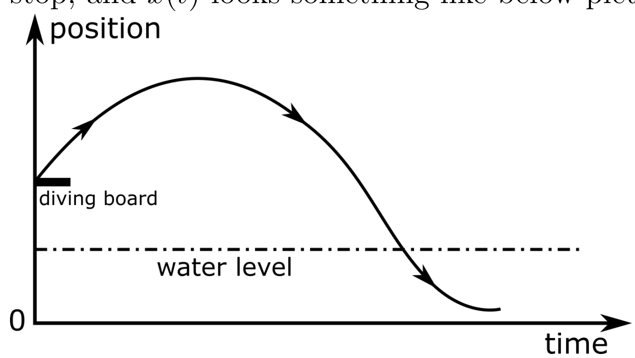
In the lecture, we covered some examples of motion graphing, as a way of helping us understand word description of a kinematics setup correctly. We will summarize these examples here, as a written reference.

A. Diver Example

The notes here are in reference to the lecture video Sketching Motion Graphs - Diver Example.

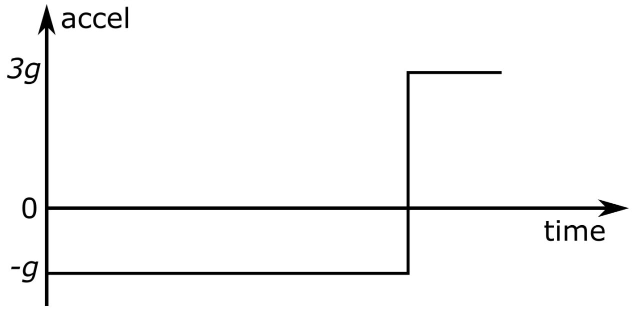
Consider a situation described this way: A diver jumps straight upward from a diving board, with some speed v_0, and on the way down, she barely misses the diving board and dives straight into the pool. And after entering the water, she slows down with an acceleration of $a = 3g$, or about 30 m/s^2.

We will graph $x(t)$, $v(t)$, and $a(t)$ motion graphs to illustrate our understanding of this setup, without reliance on memorized formulas. If you have some intuitive feel for the setup, $x(t)$ curve could be the easiest to sketch: the diver goes up and down in a "parabolic arc" that resembles a projectile trajectory, and once she goes underwater, she slows down to a stop, and $x(t)$ looks something like below picture.

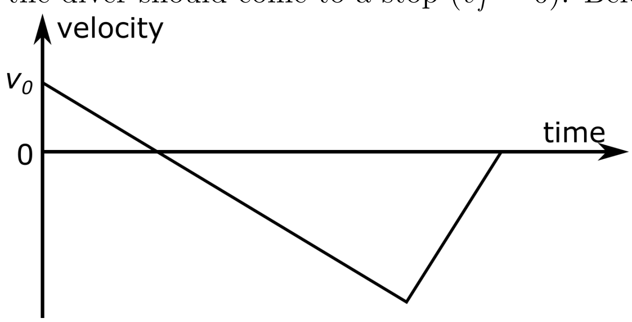


While this may be intuitively easier to sketch, not all physically relevant details are easiest to see in $x(t)$ graph. And in any case, we need to sketch $v(t)$ and $a(t)$. Of the two, it is more straightforward to sketch $a(t)$ from the given description, so we will do that next.

To graph $a(t)$, we focus on what we learned about "free fall" in kinematics. The diver's motion from leaving the diving board to entering water is a "free fall," in which only gravity is significant factor, and so she accelerates downward at g ($= 9.8$ m/s^2) for the whole time—on her way up and on her way down. The only time the diver's acceleration changes is on entering water, when acceleration reverses direction (now upward) and is at $3g$, according to description. Below sketch illustrates this $a(t)$.



Armed with this graph of $a(t)$, we can now attempt a careful sketch of $v(t)$. Since $a(t)$ represents the derivative of $v(t)$ (that is, the values of $a(t)$ give the slope of $v(t)$ at corresponding times), we need to sketch $v(t)$ with a downward slope from the start. At the time when the diver enters water, slope of $v(t)$ suddenly changes upward. And to match the setup description, the initial velocity of the diver needs to be positive ($v(0) > 0$), and at the the diver should come to a stop ($v_f = 0$). Below sketch captures all this.

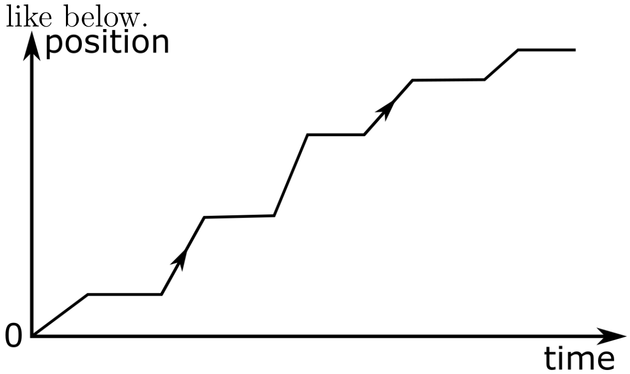


Where $v(t)$ curve crosses the time axis is around the time when the diver is at a maximum height, since at that point $dx/dt = 0$, and speed is momentarily zero. Note again that none of these motion graphs resemble each other—and they shouldn't be expected to, since they are related through derivatives ($v = dx/dt$ and $a = dv/dt$).

B. Stop-And-Go Traffic

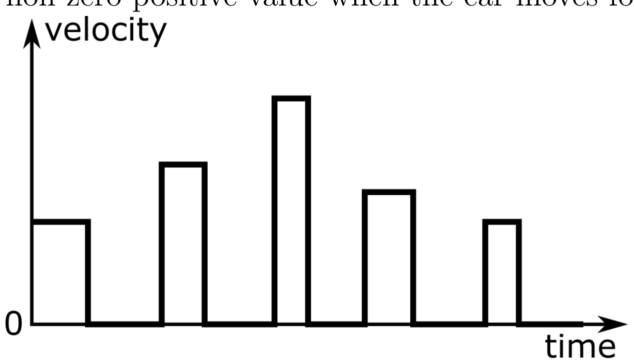
The next example is simultaneously both closer to an everyday example and it requires more abstraction and idealization to make the problem tractable. As presented in the lecture video Sketching Motion Graphs - Stop-and-Go Traffic Example, consider the motion of a car in a congested traffic condition, in a "stop-and-go" traffic. What would the motion graphs of such a car look like?

If we idealize this setup so that the car is always in two possible states of motion, a constant velocity forward or at a full stop, the position graph for the car might look something like below.

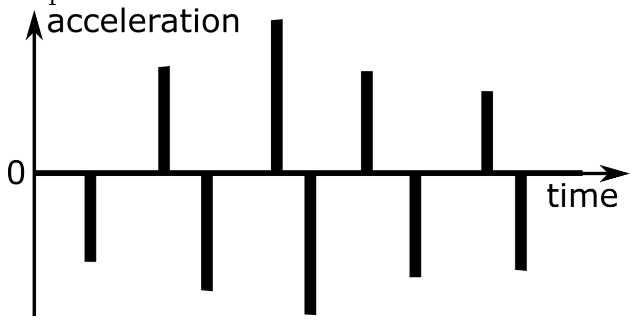


(There are other "reasonable" models one can come up with; see the end of this subsection.)

Starting with this position graph, as you sketch the velocity and acceleration graphs, you should keep in mind the fundamental relationships, $v(t) = dx/dt$, and $a(t) = dv/dt$. The values of $v(t)$ function strictly relate to the slope of the $x(t)$ function, no other superficial features. Keeping this in mind, when you finish sketching $v(t)$, it would look something like below, a piece-wise set of constant-valued functions (at zero when the car is at rest; at a non-zero positive value when the car moves forward).



The next and final graph, $a(t)$, is the least well-behaved of these functions, and this is unavoidable, once our model had us saying values of $v(t)$ has discontinuities—at these discontinuities, the derivative, $a(t)$, will end up having sharp spikes, i.e. infinities. We could say this model approximates a bad driver, who always suddenly accelerates and suddenly stops.



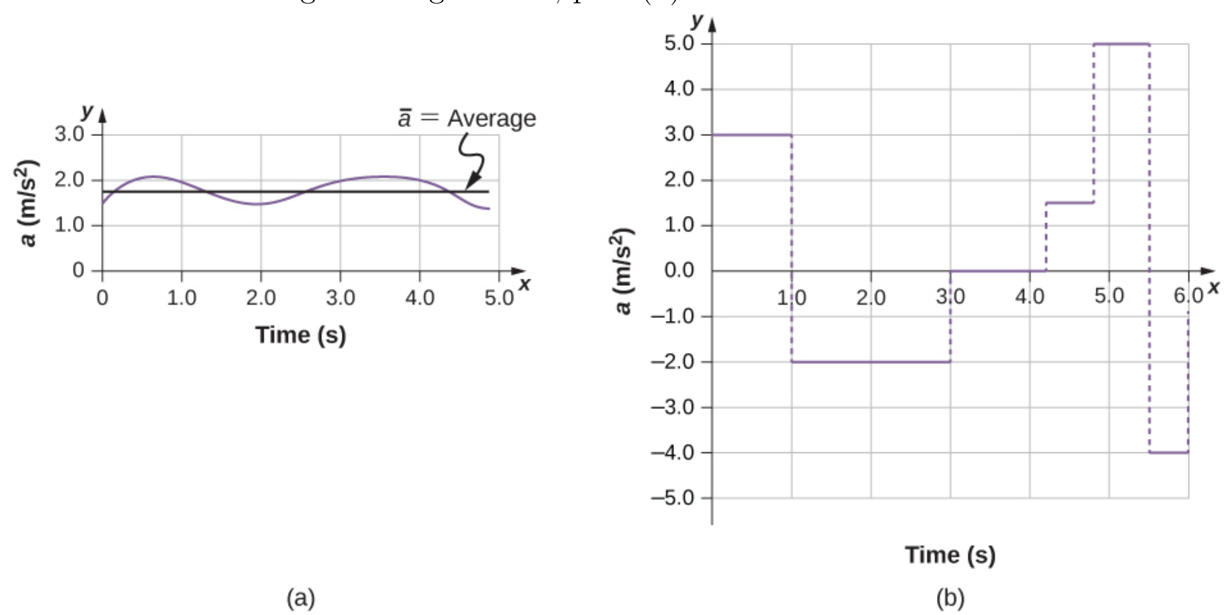
So these motion graphs illustrate the motion of a car in a "stop-and-go" traffic, with the simplifying assumptions we set out at the beginning. If you want more practice, try drawing motion graphs for the car driven by a good driver, who, in the same "stop-and-go" traffic condition, is trying to accelerate gently and brake gently (make acceleration piecewise constant functions of moderate values). How would the graphs $x(t)$, $v(t)$, and $a(t)$ look like in this case?

IV. ADDITIONAL PRACTICE

As we set out at the beginning, this is something you will benefit from looking at many examples and practicing yourself. The basic principle itself is quite simple, deriving from the two simple calculus equations $v(t) = dx/dt$ and $a(t) = dv/dt$. But there are enough pitfalls in applications that there are many common mistakes made by many students (exam problems are made out of those).

I may lecture on additional examples in the future; in the meantime, let me leave you with this invitation for additional practices.

First is one starting from Figure 3.17, part (b).



For the given $a(t)$ in (b) on right, what should the $v(t)$ and $x(t)$ look like? Does this graph give all necessary information for $v(t)$ and $x(t)$ motion graphs, or do you need to make some reasonable assumption for the situation?

The second invitation goes to different types of motion graphs that can result from constant-acceleration (i.e. $a(t) = a_0$) motion. Unfortunately the textbook sections 3.4 and 3.5 are very light on examples of motion graphs—and other examples that can benefit from introduction of motion graphs. To generate your own example, consider these possibilities: the constant acceleration a_0 can be positive or negative (2 possibilities); the initial velocity v_0 can be positive, negative, or zero (3 possibilities); the initial position x_0 can be positive, negative, or zero (3 possibilities). Non-numerically sketching these graphs (that is, getting only the general shape of the graphs and perhaps how they are positioned relative to the t-axis), how many distinct possibilities exist, and can you sketch them all correctly?